

A HODGE - DE RHAM DIRAC OPERATOR ON THE QUANTUM $SU(2)$

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ABSTRACT. We describe how it is possible to describe irreducible actions of the Hodge - de Rham Dirac operator upon the exterior algebra over the quantum spheres $SU_q(2)$ equipped with a three dimensional left covariant calculus.

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1. INTRODUCTION: THE HODGE - DE RHAM OPERATOR ON A MANIFOLD

Let M be a finite N -dimensional smooth manifold. We denote by $\Lambda(M) = \oplus_{k=0}^N \Lambda^k(M)$ the exterior algebra over it, equipped with the Cartan exterior calculus $(\Lambda(M), \wedge, d, i_X, L_X)$ with respect to the wedge product, where $d : \Lambda^k(M) \rightarrow \Lambda^{k+1}(M)$ is the exterior differential, $i_X : \Lambda^k(M) \rightarrow \Lambda^{k-1}(M)$ the contraction along the vector field X defined on M and $L_X : \Lambda^k(M) \rightarrow \Lambda^k(M)$ the Lie derivative along X . On a local chart domain with a coordinate system $\{x^i\}_{i=1,\dots,N}$ an exterior k -form is written as $\Lambda^k(M) \ni \phi = (1/k!) \phi_{a_1 \dots a_k} dx^{a_1} \wedge \dots \wedge dx^{a_k}$ with $\phi_{a_1 \dots a_k} \in \Lambda^0(M) \sim \mathcal{F}(M)$. We also denote by $\mathcal{X}(M)$ the set of vector fields over M , with $\mathcal{M} \ni X = X^a \partial_a$ along $\{\partial_a\}_{1,\dots,N}$, the local vector fields basis dual to $\{dx^a\}_{a=1,\dots,N}$, with $X^a \in \mathcal{F}(M) = \mathcal{A}$.

Let M be equipped with a metric tensor g , whose local expression is $g = g_{ab} dx^a \otimes dx^b$, or equivalently $g = g^{ab} \partial_a \otimes \partial_b$ with $g^{ab} g_{bc} = \delta_b^a$: setting¹

$$\phi \vee \phi' = \sum_s \frac{(-1)^{\binom{s}{2}}}{s!} g^{a_1 b_1} \dots g^{a_s b_s} (\tilde{\gamma}^s \{i_{a_1} \dots i_{a_s} \phi\}) \wedge \{i_{b_1} \dots i_{b_s} \phi'\}, \quad (1.1)$$

where ϕ, ϕ' are elements in $\Lambda(M)$ and $\tilde{\gamma}(\phi) = (-1)^k \phi$ for $\phi \in \Lambda^k(M)$, amounts to realise on $\Lambda(M)$ the well known Clifford product corresponding to the given metric g . It is immediate to see that

Date: 18 september 2016.

¹The reader may refer to the well known monographs [7, 9, 24, 26] for a more general description on spin structures and Dirac operators on a manifold. The book [10] and the lecture notes [23, 27] describe an algebraic setting for the study of Dirac operators in both classical and non commutative geometry.

one has

$$dx^a \vee dx^b + dx^b \vee dx^a = 2g^{ab}, \quad (1.2)$$

$$dx^a \vee dx^b = dx^a \wedge dx^b + g^{ab}. \quad (1.3)$$

The set $(\Lambda(M), \wedge, \vee, i_X, d)$ is usually called the Kähler - Atiyah algebra over (M, g) . The unital algebraic structure $(\Lambda(M), \vee)$ is the Clifford algebra $\text{Cl}(M, g)$ on M corresponding to the metric tensor g : it gives, after Kähler, an *inner* product over an exterior algebra. Notice that the Clifford algebra $\text{Cl}(M, g)$ acts upon $\Lambda(M)$, with such an action $\Phi : \Lambda(M) \rightarrow \text{End}(\Lambda(M))$ being generated by

$$\Phi(dx^a) : \phi \mapsto dx^a \wedge \phi + g^{ab} i_{\partial_b} \phi. \quad (1.4)$$

The Levi Civita connection corresponding to (M, g) is defined on $\mathfrak{X}(M)$ and dually on $\Lambda^1(M)$ via

$$\begin{aligned} \nabla : \Lambda^1(M) &\rightarrow \Lambda^1(M) \otimes_{\mathcal{A}} \Lambda^1(M), \\ \nabla : \mathcal{X}(M) &\rightarrow \Lambda^1(M) \otimes_{\mathcal{A}} \mathcal{X}(M), \end{aligned} \quad (1.5)$$

with

$$\begin{aligned} \nabla(dx^a) &= -dx^s \otimes \Gamma_{sb}^a dx^b, \\ \nabla(\partial_a) &= dx^s \otimes \Gamma_{sa}^b \partial_b \end{aligned} \quad (1.6)$$

where Γ_{bc}^a are the Christoffel symbols of the connection,

$$\Gamma_{ji}^m = \Gamma_{ij}^m = \frac{1}{2} g^{mk} (\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}). \quad (1.7)$$

The action of ∇ is extended to $\Lambda(M)$ requiring it to satisfy the Leibniz rule with respect to the wedge product. For any $\phi \in \Lambda^k(M)$, its exterior covariant derivative is defined [17] via

$$\mathfrak{D}\phi = dx^a \wedge \nabla_a \phi, \quad (1.8)$$

where $\nabla_a = \nabla_{\partial_a}$. The action of the Kähler (also called Hodge - de Rham) operator is defined as:

$$\mathcal{D}\phi = dx^a \vee \nabla_a \phi. \quad (1.9)$$

From (1.1) one immediately sees that

$$\mathcal{D}\phi = \mathfrak{D}\phi + g^{ab} i_b \nabla_a \phi. \quad (1.10)$$

A direct computation shows that $\mathfrak{D}\phi = d\phi$, so one has

$$\mathcal{D}\phi = d\phi + g^{ab} i_b \nabla_a \phi : \quad (1.11)$$

if $\phi \in \Lambda^k(M)$, the image $\mathcal{D}\phi$ has a component of degree $k+1$ and a component of degree $k-1$. Notice that $\mathcal{D}\phi$ has a well defined parity with respect to the \mathbb{Z}_2 grading of $\Lambda(M)$. One can prove that, for $\phi \in \Lambda^k(M)$,

$$\mathcal{D}\phi = d\phi + (-1)^{N(k-1)} \star d \star \phi \quad (1.12)$$

where one has introduced the Hodge duality operator corresponding to the metric g , i.e. an $\mathcal{A} = \mathcal{F}(M)$ -bimodule map $\star : \Lambda^k(M) \rightarrow \Lambda^{N-k}(M)$ defined on a basis by

$$\star(dx^{a_1} \wedge \cdots \wedge dx^{a_k}) = g^{a_1 b_1} \cdots g^{a_k b_k} i_{b_k} \cdots i_{b_1} \mu \quad (1.13)$$

where $\mu = |g|^{1/2} dx^1 \wedge \cdots \wedge dx^N$ is the invariant volume form corresponding to the metric g (here $|g|$ denotes the determinant of the matrix g_{ab}). The relation

$$\star^2(\phi) = (-1)^{k(N-k)} (\text{sgn } g) \phi \quad (1.14)$$

holds, for $\phi \in \Lambda^k(M)$, with $(\text{sgn } g)$ the signature of the metric tensor g . For riemannian (i.e. positive definite metrics g), the previous relation allows to write the action (1.12) as

$$\mathcal{D}\phi = d\phi - (-1)^k \star^{-1} d \star \phi. \quad (1.15)$$

Via a metric g on a manifold, one can introduce a scalar product among exterior forms. For homogeneous ϕ and ϕ' with the same degree we set

$$\langle \phi | \phi' \rangle = \int_M \phi \wedge \star \phi', \quad (1.16)$$

where the integration measure on M comes from the volume form μ previously defined. Homogeneous forms ϕ, ϕ' are defined orthogonal if their degrees are different (this amounts to say that $\Lambda^k(M)$ is orthogonal to $\Lambda^{k'}(M)$ for $k \neq k'$). Given any $\alpha \in \Lambda^k(M)$ and $\beta \in \Lambda^{k+1}(M)$, upon defining $d^* = (-1)^{N(k-1)+1} \star d \star : \Lambda^k(M) \rightarrow \Lambda^{k-1}(M)$ one proves that

$$\langle d\alpha | \beta \rangle = \langle \alpha | d^* \beta \rangle + \int_M d(\alpha \wedge \star \beta). \quad (1.17)$$

This relation shows that the operator d^* can then be considered – under suitable conditions on its domain – the adjoint of d in $\Lambda(M)$ with respect to the scalar product (1.16). The action (1.12) of the Dirac operator can be cast in the form

$$\mathcal{D}\phi = d\phi - d^* \phi : \quad (1.18)$$

from (1.14) we have that $d^* d^* = 0$, making it immediate to prove the following relation

$$\mathcal{D}^2 \phi = -(dd^* + d^* d)\phi = (-1)^{kN} \{\star d \star d\phi + (-1)^N d \star d \star \phi\} \quad (1.19)$$

for $\phi \in \Lambda^k(M)$. The square of the Dirac operator gives a version of the Hodge - de Rham Laplacian on $\Lambda(M)$.

It is easy to see that the action (1.4) of the Clifford algebra $\text{Cl}(M, g)$ upon $\Lambda(M)$ is highly reducible. Since the Clifford algebra $(\Lambda(M), \vee)$ is semi simple (simple for even N), its finite dimensional irreducible representations are given by its minimal (left) ideals $I \subset \Lambda(M)$, with $\Lambda(M) \vee I \subset I$. The decomposition of this algebra into minimal ideals can be characterized by a spectral set P_j of \vee -idempotents ([11]) in $(\Lambda(M), \vee)$, i.e. a set of elements satisfying:

- (1) $\sum_j P_j = 1$
- (2) $P_j \vee P_k = \delta_{jk} P_j$
- (3) the rank of P_j is minimal (non trivial), where the rank of P_j is given by the dimension of the range of the $\Lambda(M)$ -morphism $\psi \mapsto \psi \vee P_j$.

Under these conditions one has that

$$I_j = \{\psi \in \Lambda(M) : \psi \vee P_j = \psi\} \quad (1.20)$$

and $\Lambda(M) = \oplus_j I_j$. Elements in I_j are called *spinors* with respect to the idempotent P_j . The action of the Dirac operator \mathcal{D} defined in (1.12) is meaningful on a set of spinors, i.e. \mathcal{D} maps elements of I_j into elements in I_j with I_j the left ideals of the Clifford algebra $(\Lambda(M), \vee)$, if and only if the condition

$$P_j \vee \nabla_a P_j = 0 \quad (1.21)$$

for any $\nabla_a = \nabla_{\partial_a}$ holds.

The problem of defining a Clifford algebra and a Hodge - de Rham Dirac operator on quantum spaces, i.e. for non commutative algebras deforming the algebras of functions on classical spaces, has been widely studied following different approaches since for such non commutative spaces there exists no canonical definition of exterior algebra, differential calculus and symmetric tensors.

Examples of spin^c -spectral triples with a Dirac operator given by a non commutative deformation of the Dolbeault - Dirac operator (which classically coincides with the Hodge - de Rham operator

on Kähler manifolds) has been introduced and studied on quantum flag manifolds [19], quantised symmetric spaces [20, 25]: via a suitable twist a spin^c -structure on quantum complex projective spaces $\mathbb{C}P_q^N$ is seen to give a spin-structure for odd N in [3, 4].

Within the quantum group formalism, meaningful Clifford algebras and spinors are introduced in [6, 8] in terms of the properties of the R -braiding for the FRT approach. This approach is evolved in [1, 2, 14] for quantum groups equipped with a Woronowicz bicovariant exterior calculus, thus allowing for the definition of a Dirac operator. The papers [12, 13, 16] develop a consistent formulation of the notions of Clifford algebras for quantum groups equipped with left covariant Woronowicz calculi: the corresponding exterior algebras are left modules for the Clifford algebra (generalising (1.4)), spinors are introduced algebraically in terms of irreducibility subspaces of such an action. A consistent notion of metric tensors and Levi Civita connection acting upon exterior forms is introduced, a quantum analogue of the Hodge - de Rham Dirac operator generalising (1.9) is defined and studied.

The formalism developed in this series of papers is indeed consistent under conditions which are not satisfied for the case of the quantum group $\text{SU}_q(2)$ equipped with the family of three dimensional left covariant Woronowicz calculi studied in [15]. The aim of the present paper is to introduce a Hodge - de Rham Dirac operator on $\text{SU}_q(2)$ equipped with one example from that family of left invariant calculi.

In section 2 we recall the classical construction for the Hodge - de Rham operator on S^3 , and present its spectrum. In section 3, instead of defining a Clifford algebra corresponding to a suitable symmetric bilinear form, we shall introduce a scalar product among forms, a Hodge duality operator for a meaningful quantum version of the classical Cartan-Killing metric tensor, and analyse a corresponding quantum deformation of the classical expressions (1.15), (1.18). We eventually write the explicit action of a quantum Hodge - de Rham operator $\mathcal{D}_{(q)}$, and compute its spectrum.

2. THE HODGE - DE RHAM OPERATOR ON S^3

An explicit analysis of the Hodge - de Rham operator on S^3 has been given in [5]. From that paper we recall the notations and the main results. We describe the sphere S^3 as the $\text{SU}(2)$ Lie group manifold, where points are described as the entries of the matrix

$$\gamma = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}, \quad \bar{u}u + \bar{v}v = 1. \quad (2.1)$$

From a basis T_a of the Lie algebra $\mathfrak{su}(2)$, with

$$[T_a, T_b] = \epsilon_{ab}^c T_c, \quad (2.2)$$

the Maurer-Cartan 1-form

$$\gamma^{-1}d\gamma = T_a \otimes \theta^a \quad (2.3)$$

implicitly defines a left-invariant basis $\{\theta^a\}_{a=1,\dots,3}$ of $\Lambda^1(\text{SU}(2))$, whose dual vector fields are $\{L_a\}_{a=1,\dots,3}$ representing the Lie algebra structure given by $[L_a, L_b] = \epsilon_{ab}^c L_c$. They have the following explicit expression:

$$\begin{aligned} \theta^x &= i(udv - vdu + \bar{v}d\bar{u} - \bar{u}d\bar{v}), \\ \theta^y &= udv - vdu - \bar{v}d\bar{u} + \bar{u}d\bar{v}, \\ \theta^z &= 2i(\bar{u}du + \bar{v}dv) = -2i(ud\bar{u} + vd\bar{v}), \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} L_x &= \frac{i}{2} (\bar{v} \partial_u - v \partial_{\bar{u}} - \bar{u} \partial_v + u \partial_{\bar{v}}), \\ L_y &= \frac{1}{2} (-\bar{v} \partial_u - v \partial_{\bar{u}} + \bar{u} \partial_v + u \partial_{\bar{v}}), \\ L_z &= -\frac{i}{2} (u \partial_u - \bar{u} \partial_{\bar{u}} + v \partial_v - \bar{v} \partial_{\bar{v}}). \end{aligned} \quad (2.5)$$

The ladder operators

$$L_{\pm} = \frac{1}{\sqrt{2}} (L_x \pm i L_y) \quad (2.6)$$

with

$$[L_-, L_+] = i L_z, \quad [L_-, L_z] = -i L_-, \quad [L_+, L_z] = i L_+, \quad (2.7)$$

give an alternative left invariant basis, with dual 1-forms given by

$$\theta^{\pm} = \frac{1}{\sqrt{2}} (\theta^x \mp i \theta^y). \quad (2.8)$$

The differential calculus is finally characterized by

$$d\theta^a = -\frac{1}{2} f_{bc}^a \theta^b \wedge \theta^c \quad (2.9)$$

in terms of the Lie algebra structure constants along a given left invariant basis, with

$$L_a \theta^b = -f_{ac}^b \theta^c. \quad (2.10)$$

The Cartan-Killing metric tensor

$$g = \theta^x \otimes \theta^x + \theta^y \otimes \theta^y + \theta^z \otimes \theta^z = \theta^- \otimes \theta^+ + \theta^+ \otimes \theta^- + \theta^z \otimes \theta^z \quad (2.11)$$

associated to the Lie algebra structure of $\mathfrak{su}(2)$ gives a riemannian metric tensor, whose corresponding (1.13) basic Hodge duality map is, with $\tau = \theta^x \wedge \theta^y \wedge \theta^z$,

$$\star 1 = \tau, \quad \star \theta^a = -d\theta^a \quad (2.12)$$

with $\star^2 = 1$ on $\Lambda(S^3)$.

Upon the 4-dimensional set I_P of algebraic spinors with basis given by (here $a = 1, \dots, 3$)

$$\begin{aligned} w_0 &= 1 - i\tau, \\ w_a &= \delta_{ab} (\theta^b - \frac{i}{2} \varepsilon_{st}^b \theta^s \wedge \theta^t) = \delta_{ab} (\theta^b + i d\theta^b) = \delta_{ab} (\theta^b - i \star \theta^b), \end{aligned} \quad (2.13)$$

the action of the Hodge - de Rham Dirac operator $\mathcal{D} = \theta^a \vee \nabla_a : I_P \rightarrow I_P$ has the following matrix form

$$\mathcal{D}\psi = \begin{pmatrix} 0 & L_+ & L_z & L_- \\ L_- & L_z - i & -L_- & 0 \\ L_z & -L_+ & -i & L_- \\ L_+ & 0 & L_+ & -i - L_z \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_- \\ \psi_z \\ \psi_+ \end{pmatrix} \quad (2.14)$$

along the basis w_0, w_z, w_{\pm} . We consider the spectrum of the operator \mathcal{D} acting upon square integrable spinors on $SU(2)$, that is elements $\psi \in \mathbb{C}^4 \otimes \mathcal{L}^2(SU(2))$. A Peter-Weyl basis for $SU(2)$ is given by the elements

$$f_m(J, N) = (R_+)^m \{v^{J-N/2} \bar{u}^{J+N/2}\}, \quad (2.15)$$

where $N \in \mathbb{Z}$, $J = s + |N|/2$ with $s \in \mathbb{N}$, and $m = 1, \dots, 2J + 1$. This expression comes as the range of a repeated action of the ladder operators R_{\pm} (the right invariant vector fields on $SU(2)$ corresponding to the elements T_{\pm} in the Lie algebra (2.2)) upon $f_0(J, N) = v^{J-N/2} \bar{u}^{J+N/2}$. For each fixed J , one has $\mathcal{L}^2(SU(2)) \supset W_J = \{\phi : L^2 \phi = -J(J+1)\phi\}$, with $\dim W_J = (2J+1)^2$. The spectrum of \mathcal{D} is given by:

(1) for $2J \neq \pm N$, one has

$$\lambda_{\pm} = \pm i\sqrt{J(J+1)}, \quad \psi_{\pm} = \begin{pmatrix} \lambda_{\pm} f_m(J, N) \\ L_- f_m(J, N) \\ L_z f_m(J, N) \\ L_+ f_m(J, N) \end{pmatrix}, \quad (2.16)$$

$$\lambda = iJ, \quad \psi = \begin{pmatrix} 0 \\ \frac{i}{J-N/2} L_- f_m(J, N) \\ f_m(J, N) \\ -\frac{i}{J+N/2} L_+ f_m(J, N) \end{pmatrix}, \quad (2.17)$$

$$\lambda = -i(J+1), \quad \psi = \begin{pmatrix} 0 \\ -\frac{i}{1+J+N/2} L_- f_m(J, N) \\ f_m(J, N) \\ \frac{i}{1+J-N/2} L_+ f_m(J, N) \end{pmatrix}; \quad (2.18)$$

(2) for $2J = N$ one has $L_- f_m(J, 2J) = 0$, and

$$\lambda_{\pm} = \pm i\sqrt{J(J+1)}, \quad \psi_{\pm} = \begin{pmatrix} \lambda_{\pm} f_m(J, 2J) \\ 0 \\ L_z f_m(J, 2J) \\ L_+ f_m(J, 2J) \end{pmatrix}, \quad (2.19)$$

$$\lambda = -i(J+1), \quad \psi = \begin{pmatrix} 0 \\ 0 \\ -i f_m(J, 2J) \\ L_+ f_m(J, 2J) \end{pmatrix}, \quad (2.20)$$

$$\lambda = -i(J+1), \quad \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_m(J, 2J) \end{pmatrix}; \quad (2.21)$$

(3) for $2J = -N$ one has $L_+ f_m(J, -2J) = 0$ and

$$\lambda_{\pm} = \pm i\sqrt{J(J+1)}, \quad \psi_{\pm} = \begin{pmatrix} \lambda_{\pm} f_m(J, -2J) \\ L_- f_m(J, -2J) \\ L_z f_m(J, -2J) \\ 0 \end{pmatrix}, \quad (2.22)$$

$$\lambda = -i(J+1), \quad \psi = \begin{pmatrix} 0 \\ L_- f_m(J, -2J) \\ -i f_m(J, -2J) \\ 0 \end{pmatrix}, \quad (2.23)$$

$$\lambda = -i(J+1), \quad \psi = \begin{pmatrix} 0 \\ f_m(J, -2J) \\ 0 \\ 0 \end{pmatrix}. \quad (2.24)$$

By noticing that the action of \mathcal{D} commutes with the action of R_+ , we see that m is a degeneracy label for the eigenspinors corresponding to any given eigenvalue, so we have a basis for $W_j \otimes \mathbb{C}^4$, i.e. a basis for I_P made by eigenspinors for \mathcal{D} .

3. A HODGE - DE RHAM DIRAC OPERATOR ON QUANTUM SPHERES

We move to the quantum setting with a short presentation of $SU_q(2)$ and a three dimensional left covariant differential calculus on it. This presentation is based on [28, 18], more details can be found in [31]. As the quantum group $SU_q(2)$ we consider the polynomial unital $*$ -algebra $\mathcal{A}(SU_q(2)) = (SU_q(2), \Delta, S, \varepsilon)$ generated by elements a and c such that, using the matrix notation

$$u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \quad (3.1)$$

the Hopf algebra structure can be expressed as

$$\begin{aligned} uu^* &= u^*u = 1, \\ \Delta u &= u \otimes u, \\ S(u) &= u^*, \\ \varepsilon(u) &= 1. \end{aligned} \quad (3.2)$$

The deformation parameter $q \in \mathbb{R}$ is restricted without loss of generality to the interval $0 < q < 1$. By $\mathcal{U}_q(\mathfrak{su}(2))$ we denote the universal enveloping algebra of $\mathcal{A}(SU_q(2))$, generated by the elements E, F, K, K^{-1} , with Hopf $*$ -algebra structure given by

$$\begin{aligned} K^\pm E &= q^\pm EK^\pm, \\ K^\pm F &= q^\mp FK^\pm, \\ [E, F] &= \frac{K^2 - K^{-2}}{q - q^{-1}}, \end{aligned} \quad (3.3)$$

$$K^* = K, \quad E^* = F, \quad (3.4)$$

and

$$\begin{aligned} \Delta(K^\pm) &= K^\pm \otimes K^\pm, \\ \Delta(E) &= E \otimes K + K^{-1} \otimes E, \\ \Delta(F) &= F \otimes K + K^{-1} \otimes F \end{aligned} \quad (3.5)$$

for the coproduct,

$$\begin{aligned} S(K) &= K^{-1}, \\ S(E) &= -qE, \\ S(F) &= -q^{-1}F \end{aligned} \quad (3.6)$$

for the antipode and

$$\begin{aligned} \varepsilon(K) &= 1, \\ \varepsilon(E) &= \varepsilon(F) = 0 \end{aligned} \quad (3.7)$$

for the counit. The algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is a Hopf $*$ -subalgebra included in the dual algebra $\mathcal{A}(SU_q(2))^o$, which is the largest Hopf $*$ -subalgebra contained in the dual vector space $\mathcal{A}(SU_q(2))'$. The only non zero terms of the dual pairing are given upon the generators by

$$\begin{aligned} K^\pm(a) &= q^{\mp 1/2}, \quad K^\pm(a^*) = q^{\pm 1/2}, \\ E(c) &= 1, \quad F(c^*) = -q^{-1}. \end{aligned} \quad (3.8)$$

Such a pairing gives rise to the $*$ -compatible canonical commuting actions of $\mathcal{U}_q(\mathfrak{su}(2))$ upon $\mathcal{A}(SU_q(2))$

$$h \triangleright x = x_{(1)} h(x_{(2)}), \quad x \triangleleft h = x_{(2)} h(x_{(1)}) \quad (3.9)$$

for any $h \in \mathcal{U}_q(\mathfrak{su}(2))$ and $x \in \mathcal{A}(\mathrm{SU}_q(2))$.

A left covariant first order differential calculi (Γ, d) is a $\mathcal{A}(\mathrm{SU}_q(2)) = \mathcal{H}$ -bimodule provided $d : \mathcal{H} \rightarrow \Gamma$ satisfies the Leibniz rule, $d(hh') = (dh)h' + h dh'$ for $h, h' \in \mathcal{H}$, and Γ is generated by $d(\mathcal{H})$ as a \mathcal{H} -bimodule. It is called a $*$ -calculus provided there is an anti-linear involution $*$: $\Gamma \rightarrow \Gamma$ such that $(h_1(dh)h_2)^* = h_2^*(d(h^*))h_1^*$ for any $h, h_1, h_2 \in \mathcal{H}$. A first order differential calculus is said left covariant provided a left coaction $\Delta_L^{(1)} : \Gamma \rightarrow \mathcal{H} \otimes \Gamma$ exists, such that $\Delta_L^{(1)}(dh) = (1 \otimes d)\Delta(h)$ and $\Delta_L^{(1)}(h_1 \alpha h_2) = \Delta(h_1)\Delta_L^{(1)}(\alpha)\Delta(h_2)$ for any $h, h_1, h_2 \in \mathcal{H}$ and $\alpha \in \Gamma$. The set Γ turns out to be a free left covariant \mathcal{H} -bimodule, with a free basis Γ_L of left invariant one forms, namely the elements $\omega_a \in \Gamma$ such that $\Delta_L^{(1)}(\omega_a) = 1 \otimes \omega_a$. Its dimension is called the dimension of the first order calculus. A first order left covariant finite dimensional calculus is characterised by its quantum tangent space, namely a vector space $\mathcal{X} \subset \mathcal{U}_q(\mathfrak{su}(2))$, which is dual to Γ_L . A suitable basis of Γ_L gives a basis of the quantum tangent space \mathcal{X} by duality, and exact one-forms can be written as

$$dx = \sum_a (X_a \triangleright x) \omega_a \quad (3.10)$$

with $x \in \mathcal{A}(\mathrm{SU}_q(2))$.

We consider in the present paper the three dimensional differential calculus on $\mathrm{SU}_q(2)$ whose quantum tangent space \mathcal{X} has the basis

$$X_z = \frac{i}{2} \left(\frac{K^{-2} - 1}{1 - q} \right), \quad (3.11)$$

$$\begin{aligned} X_- &= -\frac{i}{\sqrt{2}} q^{-1/2} E K^{-1}, \\ X_+ &= -\frac{i}{\sqrt{2}} q^{1/2} F K^{-1}. \end{aligned} \quad (3.12)$$

with corresponding exact one forms

$$\begin{aligned} da &= -q c^* \omega_+ + a \omega_z, & dc &= a^* \omega_+ + c \omega_z, \\ da^* &= c \omega_- - q^{-1} a^* \omega_z, & dc^* &= -q^{-1} a \omega_- - q^{-1} c^* \omega_z. \end{aligned}$$

Such a differential calculus comes as a contraction of the well known $4D_+$ bicovariant differential calculus on $\mathrm{SU}_q(2)$ introduced in [29]. The basis is chosen so that antilinear hermitian conjugation upon them reads $\omega_-^* = \omega_+$, $\omega_z^* = \omega_z$.

Given the left covariant bimodule Γ over \mathcal{H} , an invertible linear mapping $\sigma : \Gamma \otimes_{\mathcal{H}} \Gamma \rightarrow \Gamma \otimes_{\mathcal{H}} \Gamma$ is called a braiding for Γ provided σ is a \mathcal{H} -bimodule homomorphism which commutes with the left coaction on Γ and satisfies the braid equation

$$(1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) = (\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \quad (3.13)$$

on $\Gamma^{\otimes 3}$. A braiding allows to define meaningful antisymmetriser operators $A^{(k)} : \Gamma^{\otimes k} \rightarrow \Gamma^{\otimes k}$. Under further suitable conditions their ranges suitably give an exterior differential algebra $(d, \Gamma_\sigma, \wedge)$, where the exterior derivative d is extended as a graded derivation with $d^2 = 0$, satisfying a graded Leibniz rule (that is $d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^m \omega \wedge d\omega'$ for any $\omega \in \Gamma_\sigma^m$).

Such a braiding neither needs to exist nor it is unique for a given left covariant differential calculus over \mathcal{H} : this is the main difference with bicovariant differential calculi, which present a canonical braiding. A non canonical braiding for the calculus we are considering, whose range gives

a well defined exterior algebra, has been introduced in [15]. It is given by

$$\begin{aligned}
\sigma(\omega_{\pm} \otimes \omega_{\pm}) &= \omega_{\pm} \otimes \omega_{\pm}, \\
\sigma(\omega_z \otimes \omega_z) &= \omega_z \otimes \omega_z + \frac{2q(q-1)}{1+q^{-1}}(\omega_+ \otimes \omega_- - \omega_- \otimes \omega_+), \\
\sigma(\omega_+ \otimes \omega_-) &= q^2 \omega_- \otimes \omega_+ + (1-q^2)\omega_+ \otimes \omega_-, & \sigma(\omega_- \otimes \omega_+) &= \omega_+ \otimes \omega_- \\
\sigma(\omega_z \otimes \omega_-) &= q^2 \omega_- \otimes \omega_z + (1-q^2)\omega_z \otimes \omega_-, & \sigma(\omega_- \otimes \omega_z) &= \omega_z \otimes \omega_- \\
\sigma(\omega_+ \otimes \omega_z) &= q^2 \omega_z \otimes \omega_+ + (1-q^2)\omega_+ \otimes \omega_z, & \sigma(\omega_z \otimes \omega_+) &= \omega_+ \otimes \omega_z.
\end{aligned} \tag{3.14}$$

Such a braiding has the following spectral decomposition,

$$(1 - \sigma)(q^2 + \sigma) = 0, \tag{3.15}$$

with

$$\begin{aligned}
\dim \ker(1 - \sigma) &= 6, \\
\dim \ker(q^2 + \sigma) &= 3.
\end{aligned} \tag{3.16}$$

The antisymmetriser operators $A^{(k)} : \Gamma^{\otimes k} \rightarrow \Gamma^{\otimes k}$ they give rise can be written as

$$\begin{aligned}
A^{(2)} &= 1 - \sigma, & \text{on } \Gamma^{\otimes 2} \\
A^{(3)} &= (1 - \sigma_2)(1 - \sigma_1 + \sigma_1 \sigma_2), & \text{on } \Gamma^{\otimes 3}
\end{aligned} \tag{3.17}$$

with $\sigma_1 = (\sigma \otimes 1)$ and $\sigma_2 = (1 \otimes \sigma)$, while $A^{(k)}$ are trivial for $k \geq 4$. A basis of left invariant two forms in Γ_{σ}^2 is given by $\{\omega_- \wedge \omega_+, \omega_+ \wedge \omega_z, \omega_z \wedge \omega_-\}$; given $\vartheta = i\omega_- \otimes \omega_+ \otimes \omega_z$, left invariant volume forms are then, up to complex numbers,

$$\theta = A^{(3)}(\vartheta). \tag{3.18}$$

The action of the antisymmetrisers $A^{(k)}$ on Γ_{σ}^k is constant. Their spectral resolution $A^{(k)}(\phi) = \lambda_{(k)}\phi$ with ϕ a k -form yields:

$$\begin{aligned}
\lambda_{(2)} &= (1 + q^2) \\
\lambda_{(3)} &= (1 + q^2)(1 + q^2 + q^4).
\end{aligned} \tag{3.19}$$

The hermitian structure over left-invariant two forms and the wedge product antisymmetry read

$$\begin{aligned}
(\omega_- \wedge \omega_+)^* &= -\omega_- \wedge \omega_+ = q^{-2}\omega_+ \wedge \omega_-, \\
(\omega_- \wedge \omega_z)^* &= -\omega_z \wedge \omega_+ = q^{-2}\omega_+ \wedge \omega_z, \\
(\omega_+ \wedge \omega_z)^* &= -\omega_z \wedge \omega_- = q^2\omega_- \wedge \omega_z,
\end{aligned}$$

while the volume form $\theta \in \Gamma_{\sigma}^3$ turns out to be a multiple of the classical one, namely the one we would obtain if the braiding were the classical flip,

$$\theta = q^2(\omega_- \otimes (\omega_+ \otimes \omega_z - \omega_z \otimes \omega_+) + \omega_+ \otimes (\omega_z \otimes \omega_- - \omega_- \otimes \omega_z) + \omega_z \otimes (\omega_- \otimes \omega_+ - \omega_+ \otimes \omega_-)). \tag{3.20}$$

If a consistent braiding is defined on a left covariant first order differential calculus, then the duality Γ_L and \mathcal{X} allows to induce a consistent braiding σ^{\dagger} on $\mathcal{X}_{\mathcal{Q}}$. If

$$\sigma(\omega_a \otimes \omega_b) = \sigma_{ab}^{ks} \omega_k \otimes \omega_s \tag{3.21}$$

then one has

$$\sigma^{\dagger}(X_a \otimes X_b) = \sigma_{ks}^{ab} X_k \otimes X_s, \tag{3.22}$$

and one can define [15] a commutator $[\cdot, \cdot] : \mathcal{X}_{\mathcal{Q}}^{\otimes 2} \rightarrow \mathcal{X}_{\mathcal{Q}}$, fulfilling a peculiar Jacobi identity on the quantum tangent space corresponding to the calculus. For the calculus we are considering it is

$$\begin{aligned} [X_a, X_a] &= 0, \\ [X_-, X_+] &= -[X_+, X_-] = -i \left(\frac{2q^2}{1+q} \right) X_z, \\ [X_z, X_-] &= -[X_-, X_z] = i \left(\frac{1+q}{2} \right) X_-, \\ [X_+, X_z] &= -[X_z, X_+] = i \left(\frac{1+q}{2} \right) X_+. \end{aligned} \quad (3.23)$$

If the quantum commutator structure $[\cdot, \cdot] : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ associated to the given braiding σ is written as $[X_a, X_b] = f_{ab}^c X_c$, one proves that a quantum version of the Maurer - Cartan formula for the differential calculus is valid, namely

$$d\omega_a = -\frac{1}{\lambda_{(2)}} f_{bc}^a \omega_b \wedge \omega_c. \quad (3.24)$$

Together with the graded Leibniz rule, this relation characterises the action of the exterior differential d upon the whole exterior algebra Γ_{σ} .

3.1. A Hodge duality on $SU_q(2)$. As in [31], we introduce a Hodge duality operator on Γ_{σ} in terms of suitable contractions, following the classical formulation that we recall.

Given a finite N -dimensional (complex, say) vector space V the action of a bilinear form $\gamma : V \times V \rightarrow \mathbb{C}$ can be extended to give (for $j \geq s$) a map $\gamma : V^{\otimes s} \times V^{\otimes j} \rightarrow V^{\otimes j-s}$ by

$$\gamma(v_{a_1} \otimes \dots \otimes v_{a_s}, v_{b_1} \otimes \dots \otimes v_{b_j}) = \gamma(v_{a_1}, v_{b_1}) \dots \gamma(v_{a_s}, v_{b_s}) v_{b_{s+1}} \otimes \dots \otimes v_{b_j} \quad (3.25)$$

If a meaningful braiding $\sigma : V^{\otimes 2} \rightarrow V^{\otimes 2}$ is defined – with $A^{(s)} : V^{\otimes s} \rightarrow V^{\otimes s}$ the corresponding antisymmetrisers – then the bilinear form γ allows to define a map over the corresponding exterior algebras, that is $\gamma : V_{\sigma}^s \times V_{\sigma}^j \rightarrow V_{\sigma}^{j-s}$ via²

$$\gamma(v_{a_1} \wedge \dots \wedge v_{a_s}, v_{b_1} \wedge \dots \wedge v_{b_j}) = \gamma(A^{(k)}(v_{a_1} \otimes \dots \otimes v_{a_k}), A^{(s)}(v_{b_1} \otimes \dots \otimes v_{b_s})). \quad (3.26)$$

It is immediate to see that, if $\theta = v_1 \wedge \dots \wedge v_N$ denotes an element in the one dimensional vector space V_{σ}^N , upon defining $\tau = t\theta$ with $\mathbb{C} \ni t \neq 0$, for the map $\gamma_{\tau} : V_{\sigma}^s \rightarrow V_{\sigma}^{N-s}$ given by $\epsilon \mapsto \gamma_{\tau}(\epsilon) = (1/s!) \gamma(\epsilon, \tau)$ one has, with $\epsilon \in V_{\sigma}^k$,

$$\gamma_{\tau}(\gamma_{\tau}(\epsilon)) = t^2 (\det_{\theta} \gamma) (-1)^{k(N-k)} \epsilon \quad \Leftrightarrow \quad \gamma(v, v') = \gamma(v', v) \quad \forall v, v' \in V, \quad (3.27)$$

where one has defined

$$\det_{\theta} \gamma = \frac{1}{N!} \gamma(\theta, \theta) :$$

the degeneracy of the map $\gamma_{\tau} \circ \gamma_{\tau}$ is related to the symmetry of the bilinear γ . Such a bilinear γ turns out to be real, that is $\gamma(v, v') \in \mathbb{R}$ for all $v^* = v$, $v'^* = v'$, if and only if, with $\tau^* = \tau$, one has $\gamma_{\tau}(\epsilon^*) = (\gamma_{\tau}(\epsilon))^*$. It is evident that, upon a suitable choice for the global normalization factor t , the map γ_{τ} is a Hodge duality on V_{σ} (the comparison with (1.13) is straightforward).

We extend this path to the quantum group setting, by considering the bilinear map $g : \Gamma_L \times \Gamma_L \rightarrow \mathbb{C}$, where Γ_L is the left invariant basis of 1-forms for the given exterior algebra $(\Gamma_{\sigma}, \wedge, d)$. Generalising (3.26), this contraction is naturally extended to the left invariant part of Γ_{σ} via

$$g(\omega_{a_1} \wedge \dots \wedge \omega_{a_k}, \omega_{b_1} \wedge \dots \wedge \omega_{b_s}) = g(A^{(k)}(\omega_{a_1} \otimes \dots \otimes \omega_{a_k}), A^{(s)}(\omega_{b_1} \otimes \dots \otimes \omega_{b_s})).$$

²We are denoting by V_{σ}^s the range of $V^{\otimes s}$ under the action of the antisymmetriser $A^{(s)}$.

From (3.18) we define the quantum volume form $\tau = \tau^* = \delta\theta$ with $\mathbb{R} \ni \delta \neq 0$ and the determinant of the contraction g

$$\det_{\theta} g = \frac{1}{\lambda_{(3)}} g(\theta, \theta). \quad (3.28)$$

We then define the linear operator $S : \Gamma_{\sigma}^k \rightarrow \Gamma_{\sigma}^{3-k}$ as

$$S(\omega) = \frac{1}{\lambda_{(k)}} g(\omega, \tau) \quad (3.29)$$

on a left-invariant basis: the action of S is extended upon all the exterior algebra Γ_{σ}^{\wedge} by requiring the left $\mathcal{A}(\text{SU}_q(2))$ -linearity. When g is non degenerate and the eigenspaces for the action of S^2 coincide with those of the action of the antisymmetrisers of the quantum differential calculus, then we say that the map g is symmetric: such map is defined real if $S(\omega^*) = (S(\omega))^*$. Under such symmetry conditions for g we define S to be a Hodge duality.

The Hodge duality allows to define [21] an hermitian left invariant inner product on Γ_{σ} . This definition is based on the notion of integral, depending on a volume, of an exterior form. One has $\int_{\tau} \omega = 0$ if $\omega \in \Gamma_{\sigma}^k$ with $k \neq 3$, while, for $\Gamma_{\sigma}^3 \ni \omega = f\tau$ with $f \in \mathcal{A}(\text{SU}_q(2))$, one sets $\int_{\tau} \omega = h(f)$ with $h : \mathcal{A}(\text{SU}_q(2)) \rightarrow \mathbb{C}$ the Haar state defined on $\text{SU}_q(2)$. In terms of this definition of integral, one sets

$$\langle \omega, \omega' \rangle = 0 \quad (3.30)$$

when ω and ω' have different degrees, while, for ω, ω' both in Γ_{σ}^k , one defines

$$\langle \omega, \omega' \rangle = \int_{\tau} \omega'^* \wedge S(\omega). \quad (3.31)$$

The inner product defined above is left invariant: one can see that

$$\langle f\omega, f'\omega' \rangle = h(f'^* f) \langle \omega, \omega' \rangle \quad (3.32)$$

for any f, f' in $\mathcal{A}(\text{SU}_q(2))$ and left invariant exterior forms ω, ω' .

Given the inner product (3.31) it is possible to define an adjoint operator to the exterior differential d , namely $d^{\dagger} : \Gamma_{\sigma}^k \rightarrow \Gamma_{\sigma}^{k-1}$ satisfying the relation

$$\langle d^{\dagger} \omega, \omega' \rangle = \langle \omega, d\omega' \rangle \quad (3.33)$$

for any $\omega, \omega' \in \Gamma_{\sigma}$. It turns out [21] that the action of the operator d^{\dagger} is

$$d^{\dagger} \omega = (-1)^k S^{-1} d(S\omega) \quad (3.34)$$

for $\omega \in \Gamma_{\sigma}^k$. Notice that, by identifying the operator S to a Hodge duality, the expression for the action of d^{\dagger} in the quantum setting coincides with that in the classical setting, see (1.15), (1.18). Notice as well that the expression (3.34) holds for any S corresponding to a non degenerate g , irrespective of the degeneracy of the spectrum of the operator S^2 .

3.2. A Cartan - Killing metric on $\text{SU}_q(2)$. Among all possible bilinear maps we select those given by (non degeneracy being equivalent to $\alpha\beta\gamma \neq 0$)

$$g(\omega_-, \omega_+) = \alpha, \quad (3.35)$$

$$g(\omega_+, \omega_-) = \beta,$$

$$g(\omega_z, \omega_z) = \gamma. \quad (3.36)$$

From [31] we know that the map S is well defined for all the three dimensional left invariant calculi \mathcal{K} on $\text{SU}_q(2)$ studied in that paper, the calculus we are considering in the present paper being one of them. In particular, the condition that the degeneracy of the spectrum of S^2 is maximal and

that the action of S commutes with the hermitian conjugation amount to linearly constrain the real variables α and β . Following [32], we set the condition (with $\mathbb{R} \ni \xi \neq 0$)

$$S(\omega^a) = \xi d\omega^a \quad (3.37)$$

for any left invariant one form ω^a – which mimics the classical condition (2.12) that characterises the Cartan - Killing metric tensor up to a global scaling – as the definition of the quantum version on $SU_q(2)$ of the Cartan - Killing metric.

From (3.20) the one has $g(\theta, \theta) = 6q^4(\alpha\beta\gamma)$. The action of the operator $S : \Gamma_\sigma^k \rightarrow \Gamma_\sigma^{3-k}$ defined in (3.29) is

$$\begin{aligned} S(\omega_-) &= i\delta\alpha\omega_z \wedge \omega_-, & S(\omega_z \wedge \omega_-) &= -2iq^4(\delta/\lambda_{(2)})\alpha\gamma\omega_- \\ S(\omega_+) &= i\delta\beta\omega_+ \wedge \omega_z, & S(\omega_+ \wedge \omega_z) &= -2iq^4(\delta/\lambda_{(2)})\beta\gamma\omega_+ \\ S(\omega_z) &= i\delta q^2\gamma\omega_- \wedge \omega_+, & S(\omega_- \wedge \omega_+) &= -2iq^2(\delta/\lambda_{(2)})\alpha\beta\omega_z \end{aligned} \quad (3.38)$$

and also

$$\begin{aligned} S(1) &= \tau, \\ S(\tau) &= \frac{6q^4}{\lambda_{(3)}}\delta^2\alpha\beta\gamma \end{aligned} \quad (3.39)$$

From the relations above it is immediate to see that $S^2(\omega_a) = \zeta\omega_a$ for any left invariant 1-form if and only if $\alpha = \beta$, and that $(S(\omega))^* = S(\omega^*)$ if and only if α, β, γ are real parameters. One sees that the Cartan - Killing metric g (among those defined in (3.35)) set by (3.37) is characterised by

$$\alpha = \beta = \left(\frac{1+q}{2q^2}\right)^2 \gamma. \quad (3.40)$$

We shall refer to the S operator defined by (3.29), (3.35), (3.40) with $\alpha \in \mathbb{R}$ as the Hodge duality operator $\star : \Gamma_\sigma^k \rightarrow \Gamma_\sigma^{3-k}$ corresponding to the *quantum* Cartan - Killing metric. We select the normalization in such a way that $S(\tau) = 1$, so that

$$\begin{aligned} \star(1) &= \tau, \\ \star(\tau) &= 1, \\ \star^2(\omega) &= A\omega, \\ \star(\omega_a) &= \xi d\omega_a \end{aligned} \quad (3.41)$$

for any 1- and 2-form ω , and any left invariant 1-form ω_a . Notice that, given the relations (3.40), one has

$$\begin{aligned} A &= \left(\frac{2}{\lambda_{(2)}}\right) \left(\frac{\lambda_{(3)}}{6}\right), \\ \xi &= -q^{-2} \left(\frac{\lambda_{(3)}}{6}\right)^{1/2} \frac{1}{\sqrt{\alpha}}. \end{aligned} \quad (3.42)$$

The coefficient A is related to the deformation of the spectrum of the quantum braiding with respect to the classical flip.

3.3. A Hodge - de Rham operator on $SU_q(2)$. The aim of this section is to prove that a suitable quantum deformation $\mathcal{D}_{(q)}$ of the action of the Hodge - de Rham operator (1.12) can be completely reduced on a subspace in Γ_σ spanned by a suitable deformation of the elements (2.13). We define the linear operator

$$\Gamma_\sigma^k \ni \omega \mapsto \mathcal{D}_{(q)}(\omega) = d\omega + \varepsilon(k) \star d(\star\omega) \quad (3.43)$$

where $\epsilon(k)$ is a complex parameter depending on the degree $k = 1, \dots, 3$. We consider the element

$$\Gamma_\sigma \ni \psi_0 = 1 + \kappa\tau : \quad (3.44)$$

it is immediate to see that the relation, with $f \in \mathcal{A}(\text{SU}_q(2))$,

$$\mathcal{D}_{(q)} : f\psi_0 \mapsto (X_a \triangleright f)\{\omega_a + \epsilon(3)\kappa(\star\omega_a)\} \quad (3.45)$$

holds. From this result we define the three elements

$$\Gamma_\sigma \ni \phi_a = \omega_a + \epsilon(3)\kappa(\star\omega_a) \quad (3.46)$$

and compute the expression $\mathcal{D}_{(q)}(f_a\phi_a)$, namely the action of $\mathcal{D}_{(q)}$ upon elements in the subspace (a left $\mathcal{A}(\text{SU}_q(2))$ -module in Γ_σ) spanned by (3.46). If we denote by $I \subset \Gamma_\sigma$ the free left $\mathcal{A}(\text{SU}_q(2))$ -module spanned by $\{\psi_0, \phi_a\}$, it is easy, using (3.41), to prove that $\mathcal{D}_{(q)}$ maps elements in I to elements in I if and only if the conditions

$$\begin{aligned} \epsilon(1) &= \epsilon(3), \\ \epsilon(2)(\epsilon(3))^2 \kappa^2 A^2 &= 1 \end{aligned} \quad (3.47)$$

hold. Since only the A parameter is fixed in the above relations, the space of solutions of such constraints is quite rich. If one wants to mimic as much as possible the classical expression, then one can choose

$$\epsilon(1) = \epsilon(3) = -\epsilon(2) = 1, \quad \kappa = \pm iA^{-1}. \quad (3.48)$$

We choose $\kappa = -iA^{-1}$. The Dirac operator (3.43) under the conditions (3.48) has a matrix representation along the basis $\{\psi_0, \phi_-, \phi_z, \phi_+\}$ on the space I given by (the action $X \triangleright f$ is defined in (3.9), with α, β, γ satisfying the constraints (3.40))

$$\mathcal{D}_{(q)} \begin{pmatrix} f_0 \\ f_- \\ f_z \\ f_+ \end{pmatrix} = \begin{pmatrix} 0 & q^4 \alpha X_+ & q^2 \gamma X_z & \beta X_- \\ X_- & A\xi^{-1}(i - \frac{2q^2}{1+q}X_z) & A\xi^{-1}\frac{2}{1+q}X_- & 0 \\ X_z & A\xi^{-1}\frac{(1+q)}{2}X_+ & A\xi^{-1}(i + (1-q)X_z) & -A\xi^{-1}\frac{1+q}{2q^2}X_- \\ X_+ & 0 & -A\xi^{-1}\frac{2q^2}{1+q}X_+ & A\xi^{-1}(i + \frac{2}{1+q}X_z) \end{pmatrix} \triangleright \begin{pmatrix} f_0 \\ f_- \\ f_z \\ f_+ \end{pmatrix} \quad (3.49)$$

The limit $q \rightarrow 1$ the action of the operator $\mathcal{D}_{(q)}$ coincides³ with the action of the operator (2.14). Our aim is now to determine the spectrum of the operator $\mathcal{D}_{(q)}$ when acting upon I , that we realise as the quantum analogue of the space $\mathbb{C}^4 \otimes \mathcal{L}^2(\text{SU}(2))$ described in the section 2, more precisely as the complex linear span along the quantum analogue of the classical Wigner functions for $\text{SU}(2)$, as showed by the Peter-Weyl theorem. We realise such a basis, related to the so called PBW construction, as

$$f_m(J, N) = (c^{J-N/2} a^{\star J+N/2}) \triangleleft E^m \quad (3.50)$$

with $J = 1/2, 1, 3/2, \dots$, $N = -2J, -2J+1, \dots, 2J-1, 2J$ and $m = 0, 1, \dots, 2J$. It is evident that the existence of such a basis directly follows from the analogy between the theory of unitary representations of the Lie group $\text{SU}(2)$ and the theory of unitary corepresentations [18] of the quantum group $\text{SU}_q(2)$.

- In order to determine the spectrum of this operator we start by considering an *Ansatz* close to the one used in (2.16) (see [5]). We consider the spinor

$$\psi = \begin{pmatrix} \sigma f_m(J, N) \\ \mu X_- \triangleright f_m(J, N) \\ \rho f_m(J, N) \\ \tilde{\mu} X_+ \triangleright f_m(J, N) \end{pmatrix}. \quad (3.51)$$

³Notice that, in the classical limit $q \rightarrow 1$, one sees from (3.42) that $A \rightarrow 1$ and $\xi \rightarrow -1$.

The eigenvalue equation $\mathcal{D}_{(q)}\psi = \lambda\psi$ upon the spinor (3.51) is equivalent to the following algebraic matrix eigenvalue equation

$$\begin{pmatrix} 0 & -\frac{\alpha}{2}q^{3-N}\phi(J, N) & \frac{2iq^5}{1+q}\alpha\left[\frac{N}{2}\right]q^{-N/2} & -\frac{\alpha}{2}q^{1-N}\epsilon(J, N) \\ 1 & -A\xi^{-1}iq^{-1}\left[\frac{N}{2}\right]q^{-N/2} & A\xi^{-1}\frac{2}{1+q} & 0 \\ \frac{i}{2}\left[\frac{N}{2}\right](1+q^{-1})q^{-N/2} & -A\xi^{-1}\frac{1+q}{4}q^{-(1+N)}\phi(J, N) & \frac{i}{2}A\xi^{-1}(1+q^{-N}) & A\xi^{-1}\frac{1+q}{4}q^{-(1+N)}\epsilon(J, N) \\ 1 & 0 & -A\xi^{-1}\frac{2q^2}{1+q} & A\xi^{-1}iq\left[\frac{N}{2}\right]q^{-N/2} \end{pmatrix}. \quad (3.52)$$

$$\cdot \begin{pmatrix} \sigma \\ \mu \\ \rho \\ \tilde{\mu} \end{pmatrix} = \lambda \begin{pmatrix} \sigma \\ \mu \\ \rho \\ \tilde{\mu} \end{pmatrix},$$

where

$$EF \triangleright f_m(J, N) = \epsilon(J, N) f_m(J, N) = \left([J + \frac{1}{2}]^2 - [\frac{1-N}{2}]^2 \right) f_m(J, N), \quad (3.53)$$

$$FE \triangleright f_m(J, N) = \phi(J, N) f_m(J, N) = \left([J + \frac{1}{2}]^2 - [\frac{N+1}{2}]^2 \right) f_m(J, N)$$

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (3.54)$$

The solutions of the eigenvalue problem for (3.52) are

$$\begin{aligned} \lambda_{\pm} &= \pm i \left(\frac{\alpha}{2} q^{1-N} \left\{ q^2 \phi(J, N) + \epsilon(J, N) + 2q^3 [\frac{N}{2}]^2 \right\} \right)^{1/2}, \\ \tilde{\mu} &= \mu, \quad \rho = \frac{i}{2} \mu [\frac{N}{2}] (1 + q^{-1}) q^{-N/2}, \quad \sigma = \mu \lambda_{\pm}, \end{aligned} \quad (3.55)$$

and

$$\begin{aligned} \lambda'_{\pm} &= \frac{i}{2} (A\xi^{-1}) \left[\frac{3 - q^{-N}}{2} \pm \sqrt{\frac{(3 - q^{-N})^2}{4} - 2 + 2q^{-N} ([J+1]^2 + [J]^2)} \right], \\ \frac{\mu}{\rho} &= \frac{(\lambda_{\pm} - iA\xi^{-1}q^{1-N/2}[\frac{N}{2}])}{(\lambda_{\pm} + iA\xi^{-1}q^{-1-N/2}[\frac{N}{2}])} \frac{\tilde{\mu}}{\rho} + \frac{2A\xi^{-1}(1+q^2)}{(1+q)(\lambda_{\pm} + iA\xi^{-1}q^{-1-N/2}[\frac{N}{2}])}, \\ \frac{\sigma}{\rho} &= \left(\lambda_{\pm} - iA\xi^{-1}q^{1-N/2}[\frac{N}{2}] \right) \frac{\tilde{\mu}}{\rho} + \frac{2A\xi^{-1}q^2}{1+q}, \\ \frac{\tilde{\mu}}{\rho} &= \{ 2\lambda^2 A\xi^{-1} \frac{q^2}{1+q} + \frac{2iq^{1-N/2}}{1+q} [\frac{N}{2}] ((A\xi^{-1})^2 - q^4 \alpha) \lambda + \alpha \frac{q^{3-N}(A\xi^{-1})}{1+q} (1 + q^2 + 2q [\frac{N}{2}]) \} \cdot \\ &\quad \left\{ -\lambda_{\pm}^3 - i\lambda_{\pm}^2 A\xi^{-1} [\frac{N}{2}] q^{-N/2} (q - q^{-1}) - \lambda_{\pm} (\frac{\alpha}{2} (q^{3-N} \phi + q^{1-N} \epsilon) + (A\xi^{-1})^2 q^{-N} [\frac{N}{2}]^2) + \right. \\ &\quad \left. iA\xi^{-1} \frac{\alpha}{2} q^{-3N/2} [\frac{N}{2}] (q^4 \phi - \epsilon) \right\}^{-1}. \end{aligned}$$

- For $2J = N$ one has $X_{-} \triangleright f_m(J = N/2, N) = 0$. We explore the eigenvalue problem upon considering the ansatz

$$\psi = \begin{pmatrix} \sigma f_m(J = N/2, N) \\ 0 \\ \rho f_m(J = N/2, N) \\ \tilde{\mu} X_{+} \triangleright f_m(J = N/2, N) \end{pmatrix} \quad (3.56)$$

for an eigenspinor of $\mathcal{D}_{(q)}$. The action of $\mathcal{D}_{(q)}$ is meaningful upon such a space of spinors, and the eigenvalue problem for $\mathcal{D}_{(q)}$ is equivalent to the following matrix eigenvalue problem

$$\begin{pmatrix} 0 & \frac{i}{2} q^2 \gamma \frac{q^{-N}-1}{1-q} & -\frac{1}{2} \beta q^{1-N} [N] \\ \frac{i}{2} \left(\frac{q^{-N}-1}{1-q} \right) & A \xi^{-1} \frac{i}{2} (1+q^{-N}) & A \xi^{-1} \frac{1}{4} q^{-1-N} (1+q) [N] \\ 1 & -A \xi^{-1} \frac{2q^2}{1+q} & i A \xi^{-1} q^2 \left(\frac{q^{-N}-1}{1-q^2} \right) \end{pmatrix} \begin{pmatrix} \sigma \\ \rho \\ \tilde{\mu} \end{pmatrix} = \lambda \begin{pmatrix} \sigma \\ \rho \\ \tilde{\mu} \end{pmatrix} \quad (3.57)$$

The first set of eigenvalues and corresponding eigenspinors is given by

$$\begin{aligned} \lambda_{\pm} &= \pm i \left(\frac{\alpha}{2} q^{1-N} \left\{ q^2 \phi(J = N/2, N) + \epsilon(J = N/2, N) + 2q^3 \left[\frac{N}{2} \right]^2 \right\} \right)^{1/2} \\ &= \pm i \left(\frac{\alpha}{2} q^{1-2J} \{ [2J] + 2q^3 [J]^2 \} \right)^{1/2}, \\ \tilde{\mu} &\neq 0, \quad \rho = \frac{i}{2} \mu \left[\frac{N}{2} \right] (1+q^{-1}) q^{-N/2}, \quad \sigma = \tilde{\mu} \lambda_{\pm}. \end{aligned} \quad (3.58)$$

Such eigenvalues coincide, for $N = 2J$, with the eigenvalues (3.55): the parameter ρ turns indeed out to be given by $X_z \triangleright a^{\star N} = \rho a^{\star N}$. This allows to write the eigenspinors corresponding to the eigenvalues (3.58) as

$$\psi = \begin{pmatrix} \lambda_{\pm} f \\ 0 \\ X_z \triangleright f \\ X_{+} \triangleright f \end{pmatrix} \quad (3.59)$$

con $f = f_m(J = N/2, N)$.

The third eigenvalue of the matrix (3.57) is given by

$$\lambda = i A \xi^{-1} \left(\frac{1}{2} (1+q^{-2J}) + q^2 \left(\frac{q^{-2J}-1}{1-q^2} \right) \right), \quad (3.60)$$

its corresponding eigenspinor is given by

$$\begin{aligned} \sigma &= \frac{i \sqrt{\frac{3}{2}} \sqrt{\alpha} (q^2 + 1) \sqrt{\frac{q^4 + q^2 + 1}{q^2 + 1}} q^{-N} (q^{2N} - 1)}{5q^{N+2} + 3q^{N+4} + 4q^N - q^4 - 3q^2 - 2} \\ \rho &= - \frac{i(q+1)q^{-N-4} (3q^{2N} + 2q^{N+2} + 2q^{N+4} + 2q^{N+6} + 7q^{2N+2} + 5q^{2N+4} + 3q^{2N+6} - q^6 - 3q^4 - 5q^2 - 3)}{4(5q^{N+2} + 3q^{N+4} + 4q^N - q^4 - 3q^2 - 2)} \\ \tilde{\mu} &= 1 \end{aligned} \quad (3.61)$$

Once fixed $2J = N$, a direct inspection shows that the action of the operator $\mathcal{D}_{(q)}$ is diagonal upon the pair eigenvalue-eigenspinor given by

$$\lambda = i A \xi^{-1} q^{-J} [1 + J], \quad \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_m(J = N/2, N) \end{pmatrix} \quad (3.62)$$

- We now consider the case $N = -2J$, with $X_{+} \triangleright f_m(J = -N/2, N) = 0$, and consider for an eigenspinor the ansatz

$$\psi = \begin{pmatrix} \sigma f_m(J = -N/2, N) \\ \mu X_{-} \triangleright f_m(J = -N/2, N) \\ \rho f_m(J = -N/2, N) \\ 0 \end{pmatrix}. \quad (3.63)$$

The eigenvalue problem for the action of $\mathcal{D}_{(q)}$ is meaningful upon such a space of spinors, and turns out to be equivalent to the matrix eigenvalue problem given by

$$\begin{pmatrix} 0 & -\frac{1}{2}\alpha q^{3+2J}[2J] & q^2 \gamma \frac{i}{2} \left(\frac{q^{2J}-1}{1-q} \right) \\ 1 & iA\xi^{-1} \left(\frac{1-q^{2J}}{1-q^2} \right) & A\xi^{-1} \frac{2}{1+q} \\ \frac{i}{2} \left(\frac{q^{2J}-1}{1-q} \right) & -\frac{1}{4}A\xi^{-1}(1+q)q^{2J-1}[2J] & A\xi^{-1} \frac{i}{2}(1+q^{2J}) \end{pmatrix} \begin{pmatrix} \sigma \\ \mu \\ \rho \end{pmatrix} = \lambda \begin{pmatrix} \sigma \\ \mu \\ \rho \end{pmatrix} \quad (3.64)$$

The first set of eigenvalues and corresponding eigenvectors for this matrix is

$$\lambda_{\pm} = \pm i \left(\frac{\alpha}{2} q^{1+2J} \{q^2[2J] + 2q^3[J]^2\} \right)^{1/2},$$

$$\mu \neq 0, \quad \rho = -\frac{i}{2} \mu [J](1+q^{-1})q^J, \quad \sigma = \mu \lambda_{\pm}. \quad (3.65)$$

The third eigenvalue of (3.64) is

$$\lambda = iA\xi^{-1} \left(\frac{1}{2}(1+q^{2J}) + \left(\frac{1-q^{2J}}{1-q^2} \right) \right), \quad (3.66)$$

with corresponding eigenspinor

$$\sigma = \frac{2\sqrt{6}\sqrt{\alpha}q^2 (q^2 + 1) \sqrt{\frac{q^4+q^2+1}{q^2+1}} (q^{2J} - 1)}{(q+1)(3q^{2J+2} + q^{2J+4} + 2q^{2J} + q^4 - q^2)} \quad (3.67)$$

$$\mu = -\frac{4i(q^{2J+4} - q^{2J} + q^4 + 2q^2 + 3)}{(q+1)(2q^{4J} + 2q^{2J+2} + 2q^{2J+4} + 3q^{4J+2} + q^{4J+4} + 2q^{2J} + q^4 - q^2)}$$

$$\rho = 1.$$

Analogously to (3.62), the action of $\mathcal{D}_{(q)}$ is diagonal upon the pair eigenvalue-eigenspinor given by

$$\lambda = iA\xi^{-1} \{1 + q^{J+1}[J]\}, \quad \psi = \begin{pmatrix} 0 \\ f_m(J = -N/2, N) \\ 0 \\ 0 \end{pmatrix} \quad (3.68)$$

The spectrum of $\mathcal{D}_{(q)}$ turns out, by analysing the classical limit $q \rightarrow 1$, to be a quantum deformation of the spectrum of the Hodge - de Rham operator studied in [5] and described in the section 2. Such a spectrum depends not only on J , as in the classical setting, but also on N . This is a well known phenomenon in the theory of differential operators on quantum spaces: quantising a classical differential operator amounts to remove some of the degeneracies of its classical spectrum.

Upon explicitly counting the multiplicities of the the eigenvalues written above (notice that, as obvious since the action of $\mathcal{D}_{(q)}$ is written in terms of only left acting operators, its spectrum does not depend on m), one can, as already pointed out in section 2 for the classical counterpart of $\mathcal{D}_{(q)}$, say that we have explicitly written a basis for I made of eigenspinors for $\mathcal{D}_{(q)}$.

We also mention that we equipped the quantum group $SU_q(2)$ with a well known left covariant three dimensional calculus, so that all the computations related to the explicit action of the Dirac

operator and its spectrum strongly depends on it. We leave to a further analysis to present Hodge - de Rham Dirac operators on the same quantum group $SU_q(2)$ equipped with the family of three dimensional left covariant calculi studied in [31].

Acknowledgements. We should like to thank Patrizia Vitale and Franco Ventriglia for many interesting discussions. We gratefully acknowledge the support of the INFN, of the Spanish MINECO grant MTM2014-54692-P and Quitemad+, S2013/ICE-2801, of the University of Luxembourg. We worked at this paper while visiting the ICMAT (Madrid), the Center for High Energy Physics (Bangalore), the Brookhaven National Laboratory (Upton, NY): we are happy to thank our hosts there.

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